Differential Invariants of the Kerr Vacuum

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The norms associated with the gradients of the two non-differential invariants of the Kerr vacuum are examined. Whereas both locally single out the horizons, their global behavior is more interesting. Both reflect the background angular momentum as the volume of space allowing a timelike gradient decreases with increasing angular momentum becoming zero in the degenerate and naked cases. These results extend directly to the Kerr-Newman geometry.

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Recently [1] I reviewed, in the Kerr vacuum, the two independent invariants derivable from the Weyl tensor (and its dual) without differentiation, and showed that both of these non-differential invariants must be examined in order to avoid an erroneous conclusion that the ring singularity of this spacetime is, in any sense, "directional". In this companion piece I examine differential invariants with one differentiation (that is, invariants containing no more than three derivatives of the metric tensor). After reviewing invariants constructed directly from covariant derivatives of the Weyl tensor (and its dual), invariants which appear to have little if any significance, I explore the norms associated with the gradients of the two non-differential invariants. These invariants do single out the horizons, but their global behavior is more interesting as they reflect the background angular momentum in the following way: the volume of space allowing a timelike gradient decreases with increasing angular momentum becoming zero in the degenerate case. The nakedly singular cases allow no timelike gradients. Important multimedia enhancements to this work are at `http://grtensor.org/diffweyl/`

We start by reviewing invariants constructed directly from covariant derivatives of the Weyl tensor. In terms of the familiar Boyer-Lindquist coordinates [2], write $x \equiv r/a$ and $A \equiv m/a > 0$, both with $a \neq 0$, and write C_{ijkl} as the Weyl tensor with $\bar{C}_{ijkl}$ its dual. It follows that $\nabla_m C_{ijkl} \nabla^m C^{ijkl} = -\nabla_m \bar{C}_{ijkl} \nabla^m \bar{C}^{ijkl}$ and so there are two quadratic invariants derivable from covariant derivatives of the Weyl tensor. Write these as $\mathcal{D}_s \equiv \nabla_m C_{ijkl} \nabla^m \bar{C}^{ijkl} m^6 A^8/5760$ and $\mathcal{D} \equiv \nabla_m C_{ijkl} \nabla^m C^{ijkl} m^6 A^8/720$. It follows that

$$\mathcal{D}_s = -\frac{x x1\,(x-x1)(x+x1)(x-x2)(x+x2)(x-x3)(x+x3)f(x)}{(x^2+x1^2)^9}, \tag{1}$$

and

$$\mathcal{D} = \frac{(x-x6)(x+x6)(x-x7)(x+x7)(x-x8)(x+x8)(x-x9)(x+x9)f(x)}{(x^2+x1^2)^9}, \tag{2}$$

where

$$x1 = \cos(\theta), \quad x2 = (1-\sqrt{2})x1, \quad x3 = (1+\sqrt{2})x1, \tag{3}$$

and

$$f(x) \equiv x^2 - 2\,\frac{x}{A} + x1^2, \tag{4}$$

which allows the restricted factorization

$$(x-x4)(x-x5), \tag{5}$$

with

$$x4 = \frac{1+\sqrt{1-(Ax1)^2}}{A}, \quad x5 = \frac{1-\sqrt{1-(Ax1)^2}}{A}, \tag{6}$$

for $(Ax1)^2 \leq 1$,

$$x6 = (-1-\sqrt{2}+\sqrt{4+2\sqrt{2}})x1, \quad x7 = (1-\sqrt{2}+\sqrt{4-2\sqrt{2}})x1, \tag{7}$$

and

$$x8 = (-1+\sqrt{2}+\sqrt{4-2\sqrt{2}})x1, \quad x9 = (1+\sqrt{2}+\sqrt{4+2\sqrt{2}})x1. \tag{8}$$

The invariant $\mathcal{D}$ was first discussed by Karlhede et al [3] over two decades ago, but an unfortunate factorization in that paper would lead to the suggestion that $\mathcal{D}$ vanishes only on the ergo surfaces $x4$ and $x5$. (An extensive discussion of $x4$ is given in [4].) The roots for $\mathcal{D}$ have been given previously by Gass et al [5]. The significance of $\mathcal{D}_s$, if any, is unclear. The significance of $\mathcal{D}$ also remains unclear except for the fact that in the equatorial plane ($x1 = 0, x > 0$) it vanishes only on the outer ergo surface ($x4$, $x = 2/A$). What is clear is that in the limit $x \to 0$, $\theta \to \pi/2$ ($\equiv \mathcal{S}$) both $\mathcal{D}_s$ and $\mathcal{D}$ remain zero along the inner ergo surface ($x = x5 \to 0$). Three dimensional images of $\mathcal{D}_s$ and $\mathcal{D}$ are available at the web site.

The Kerr vacuum has two independent invariants derivable from the Riemann tensor without differentiation [1]. The norms associated with the gradients of these scalar fields form a rather natural class of differential invariant for investigation and yet there appears to be no discussion of these in the literature. These gradients are defined by $v_e \equiv \nabla_e(C_{ijkl}C^{ijkl}) = -\nabla_e(\bar{C}_{ijkl}\bar{C}^{ijkl})$ and $w_e \equiv \nabla_e(C_{ijkl}\bar{C}^{ijkl}) = \nabla_e(\bar{C}_{ijkl}C^{ijkl})$. For v a direct calculation gives

$$v_r = -288\,\frac{x(-7\,(x1)^2(-5\,(x1)^2x^2+(x1)^4+3\,x^4)+x^6)}{m^5A^7(x^2+(x1)^2)^7}, \quad v^r = \frac{x^2A-2\,x+A}{A(x^2+x1^2)}v_r, \tag{9}$$

$$v_\theta = -288\,\frac{x1\,(-(x1)^2(-21\,(x1)^2x^2+(x1)^4+35\,x^4)+7\,x^6)}{m^5A^7(x^2+(x1)^2)^7}, \quad v^\theta = \frac{1-x1^2}{x^2+x1^2}v_\theta, \tag{10}$$

with $v_\phi = v^\phi = v_t = v^t = 0$. Similarly for w, $w_r = -v_\theta$, $w_\theta = v_r$, $w^r = (x^2A - 2x + A)/(A(x^2 + x1^2))w_r$, $w^\theta = (1 - x1^2)/(x^2 + x1^2)w_\theta$ and $w_\phi = w^\phi = w_t = w^t = 0$.

The scaled norm $\mathcal{V} \equiv (v_ev^e)A^{15}m^{10}/288^2$ is given by

$$\begin{aligned}\mathcal{V} = &-(-x^{14}(x^2A-2\,x+A)+x1^2(Ax1^{14}-42\,x^2Ax1^{12}-x1^{12}A+462\,x^4Ax1^{10}+98\,x^3x1^{10}-7\,x^2x1^{10}A\\ &-994\,x^6Ax1^8-980\,x^5x1^8-21\,x^4x1^8A+3038\,x^7x1^6-35\,x^6x1^6A+994\,x^{10}Ax1^4-2968\,x^9x1^4-35\,x^8x1^4A\\ &-462\,x^{12}Ax1^2+1022\,x^{11}x1^2-21\,x^{10}x1^2A+42\,x^{14}A-84\,x^{13}-7\,x^{12}A))\\ &/(x^2+x1^2)^{15}.\end{aligned} \tag{11}$$

Along $x = 0$

$$\mathcal{V} = -\frac{A(x1-1)(x1+1)}{x1^{16}}. \tag{12}$$

In the equatorial plane $x1 = 0$

$$\mathcal{V} = \frac{x^2A-2\,x+A}{x^{16}}, \tag{13}$$

and along the axis $x1^2 = 1$

$$\mathcal{V} = \frac{x^2(x^6-21\,x^4+35\,x^2-7)^2(x^2A-2\,x+A)}{(x^2+1)^{15}}. \tag{14}$$

For $0 < A \leq 1$, $x^2A - 2\,x + A = 0$ at the horizons

$$x10 = \frac{1+\sqrt{1-A^2}}{A}, \quad x11 = \frac{1-\sqrt{1-A^2}}{A}. \tag{15}$$

The real positive roots to $x^6 - 21x^4 + 35x^2 - 7 = 0$ are $x \simeq 0.4815746188, 1.253960338, 4.381286268$. Unlike $\mathcal{D}$ or $\mathcal{D}_s$, $\mathcal{V}$ shows a strong sign dependence with A. Whereas the vanishing of $\mathcal{V}$ in the equatorial plane (and on the axis away from the discrete roots given) obviously singles out the horizons, the global behavior of $\mathcal{V}$ is rather more interesting than that. In general terms, the area of regions where $\mathcal{V} < 0$ (and so v is timelike) on any $r - \theta$ hypersurface decreases as $A \to 1^-$ and vanishes for $A \geq 1$. This is shown in Figures 1 through 3 where plots of $\mathcal{V}$ are shown truncated at

$\mathcal{V} = 0, +1$ for $A = 1/2, 0.95, 1$. An animation of this evolution is at the web site. A plot of $\mathcal{V} = 0$ for A close to 1 is shown in Figure 4.

The scaled norm $\mathcal{W} \equiv (w_e w^e) A^{15} m^{10}/288^2$ is given by

$$\begin{aligned}\mathcal{W} = (&Ax^{14} + 48\,Ax1^2x^{14} - 98\,x1^2x^{13} - 448\,Ax1^4x^{12} + 7\,Ax1^2x^{12} + 980\,x1^4x^{11} + 21\,Ax1^4x^{10} + 1008\,Ax1^6x^{10} \\ &-3038\,x1^6x^9 + 35\,Ax1^6x^8 + 2968\,x1^8x^7 - 1008\,Ax1^{10}x^6 + 35\,Ax1^8x^6 - 1022\,x1^{10}x^5 + 21\,Ax1^{10}x^4 \\ &+448\,Ax1^{12}x^4 + 84\,x1^{12}x^3 + 7\,Ax1^{12}x^2 - 48\,Ax1^{14}x^2 - 2\,x1^{14}x + Ax1^{14}) \\ &/(x^2 + x1^2)^{15}.\end{aligned} \tag{16}$$

Along $x = 0$

$$\mathcal{W} = \frac{A}{x1^{16}}. \tag{17}$$

In the equatorial plane $x1 = 0$

$$\mathcal{W} = \frac{A}{x^{16}}, \tag{18}$$

and along the axis $x1^2 = 1$

$$\mathcal{W} = \frac{(7\,x^6 - 35\,x^4 + 21\,x^2 - 1)^2(Ax^2 - 2\,x + A)}{(x^2 + 1)^{15}}. \tag{19}$$

The real positive roots to $7\,x^6 - 35\,x^4 + 21\,x^2 - 1 = 0$ are $x \simeq 0.2282434744, 0.7974733889, 2.076521397$. Again, unlike $\mathcal{D}$ or $\mathcal{D}_s$, $\mathcal{W}$ shows a strong sign dependence with A. Whereas the vanishing of $\mathcal{W}$ on the axis obviously singles out the horizons (away from the discrete roots given), the global behavior of $\mathcal{W}$, like $\mathcal{V}$, is rather more interesting than that. In general terms, the area of regions where $\mathcal{W} < 0$ (and so w is timelike) on any $r - \theta$ hypersurface decreases as $A \to 1^-$ and vanishes for $A \geq 1$ in a manner similar to $\mathcal{V}$. This is shown in Figures 5 through 7 where plots of $\mathcal{W}$ are shown truncated at $\mathcal{W} = 0, +1$ for $A = 1/2, 0.95, 1$. An animation of this evolution is at the web site. A plot of $\mathcal{W} = 0$ for A close to 1 is shown in Figure 8.

For completeness we note that the scaled norm $\mathcal{X} \equiv (v_e w^e) A^{16} m^{10}/288^2$ is given by

$$\mathcal{X} = -\frac{xx1\,(x^6 - 21\,x^4x1^2 + 35\,x^2x1^4 - 7\,x1^6)(7\,x^6 - 35\,x^4x1^2 + 21\,x^2x1^4 - x1^6)f(x)}{(x^2 + x1^2)^{15}}. \tag{20}$$

$\mathcal{X}$ shares none of the interesting global properties of $\mathcal{V}$ or $\mathcal{W}$.

The norms associated with the gradients of the two non-differential invariants of the Kerr vacuum have been examined. It has been shown that both locally single out the horizons, but more importantly, their global behavior reflects the background angular momentum as the volume of space allowing a timelike gradient decreases with increasing angular momentum, becoming zero in the degenerate and naked cases. One would certainly like to know if these results are robust as regards an extension to the Kerr-Newman geometry. In a subsequent note [6] I show that indeed they are.

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[6] K. Lake,"Differential Invariants of the Kerr-Newman geometry" (in preparation).
[7] This is a package which runs within Maple. It is entirely distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address *http://grtensor.org*

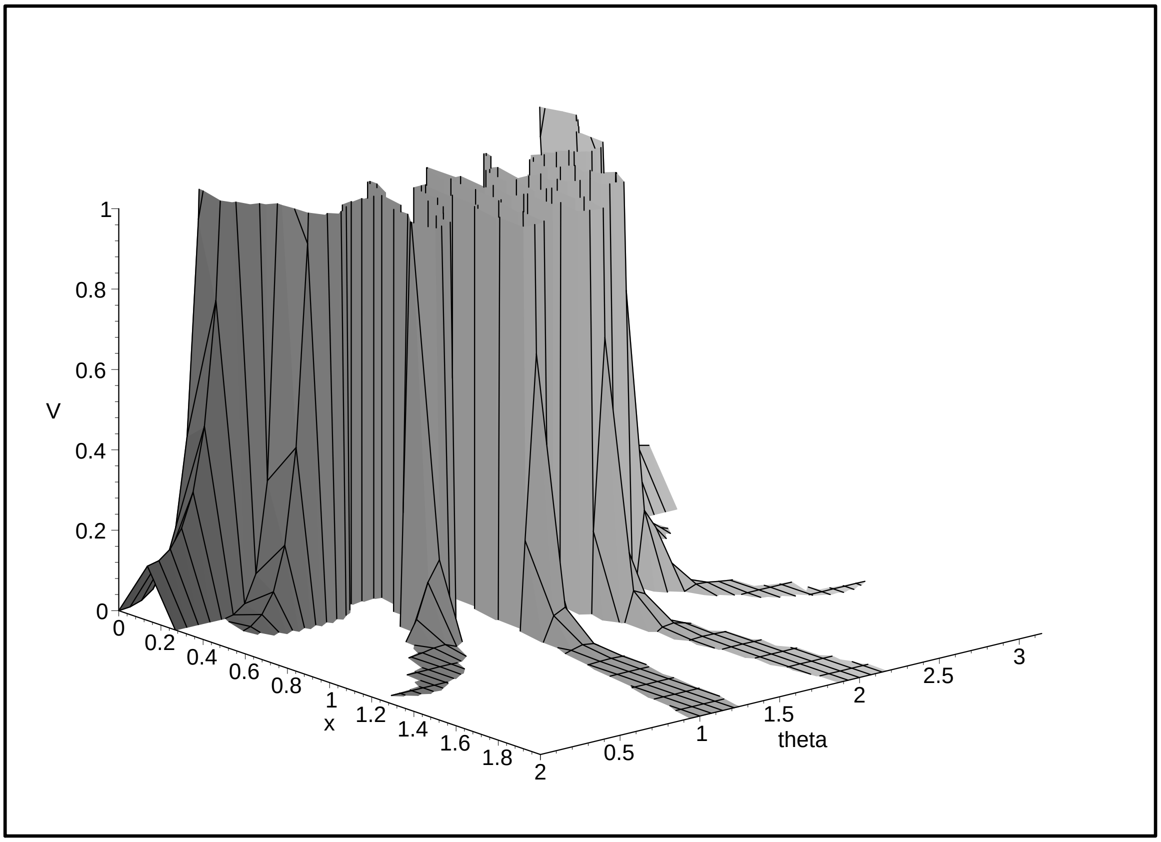

FIG. 1: Plot of $\mathcal{V}$ truncated at $\mathcal{V} = 0, +1$ for $A = 1/2$.

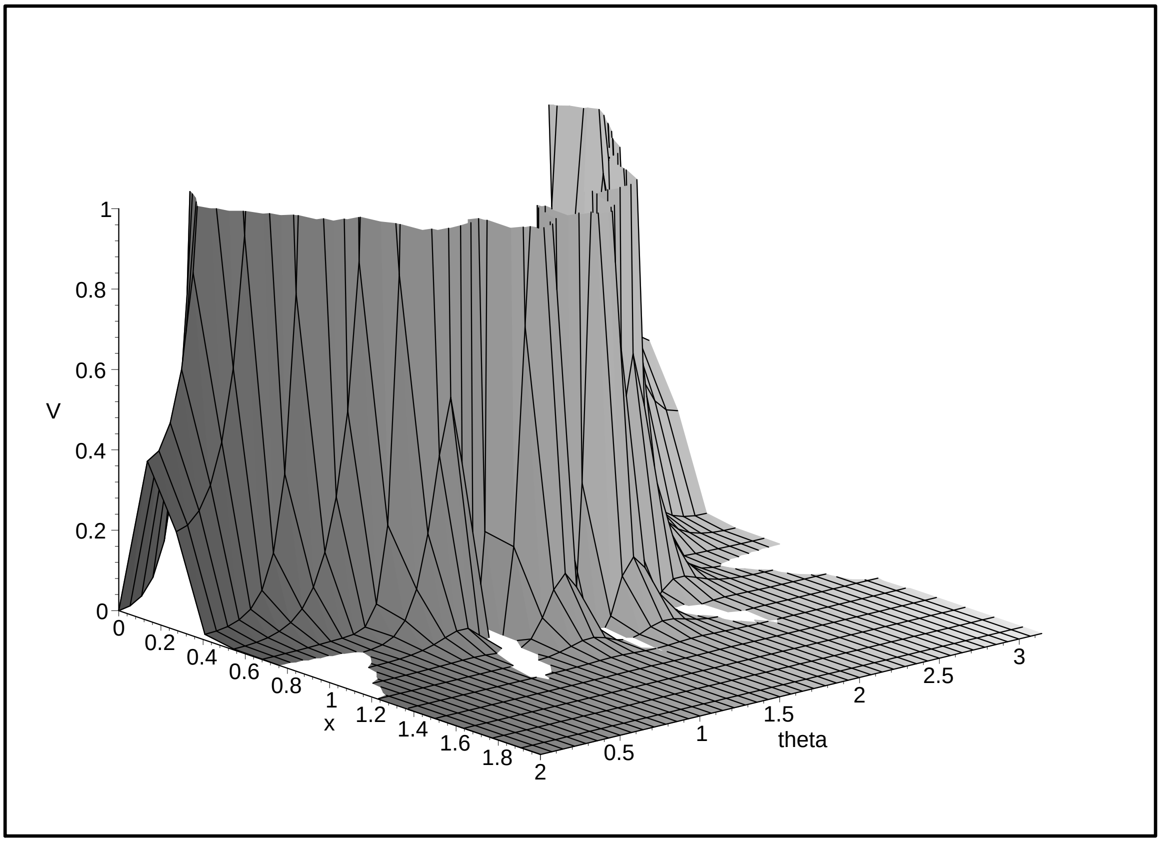

FIG. 2: Plot of $\mathcal{V}$ truncated at $\mathcal{V} = 0, +1$ for $A = 0.95$.

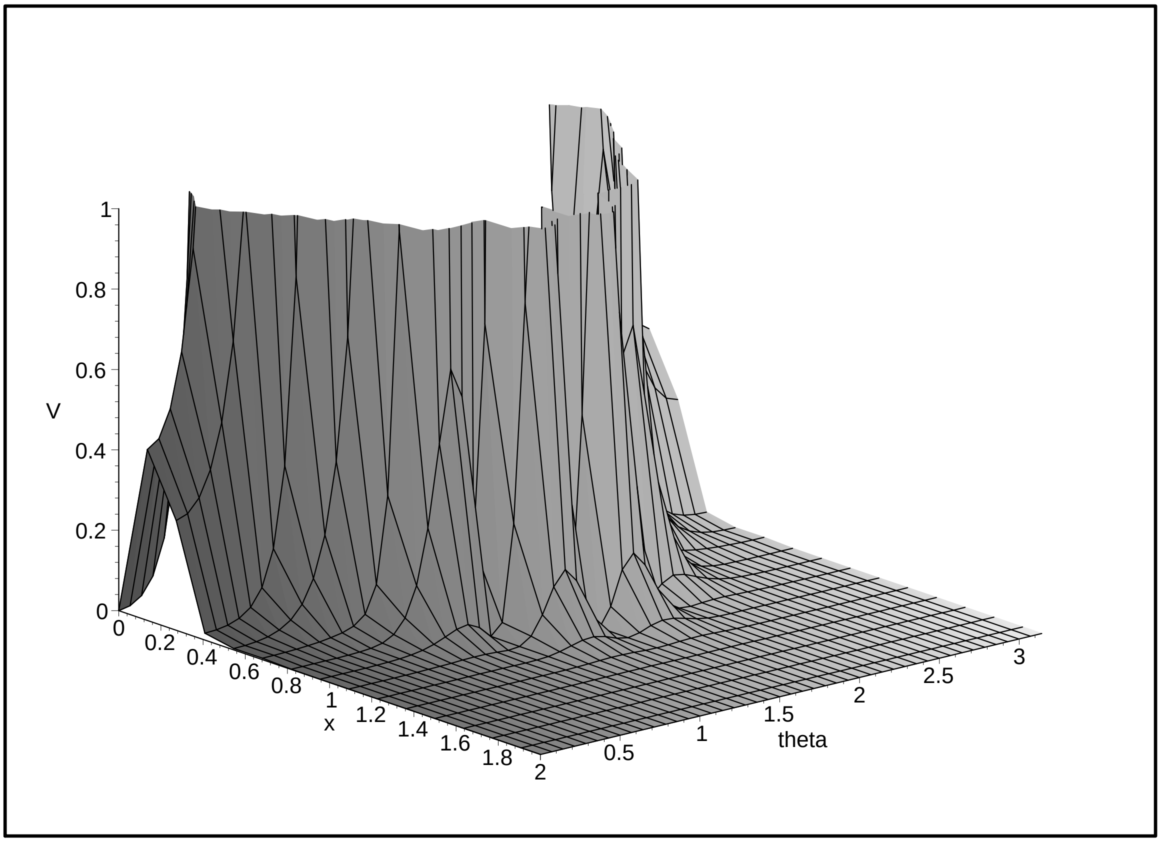

FIG. 3: Plot of $\mathcal{V}$ truncated at $\mathcal{V} = 0, +1$ for $A = 1$.

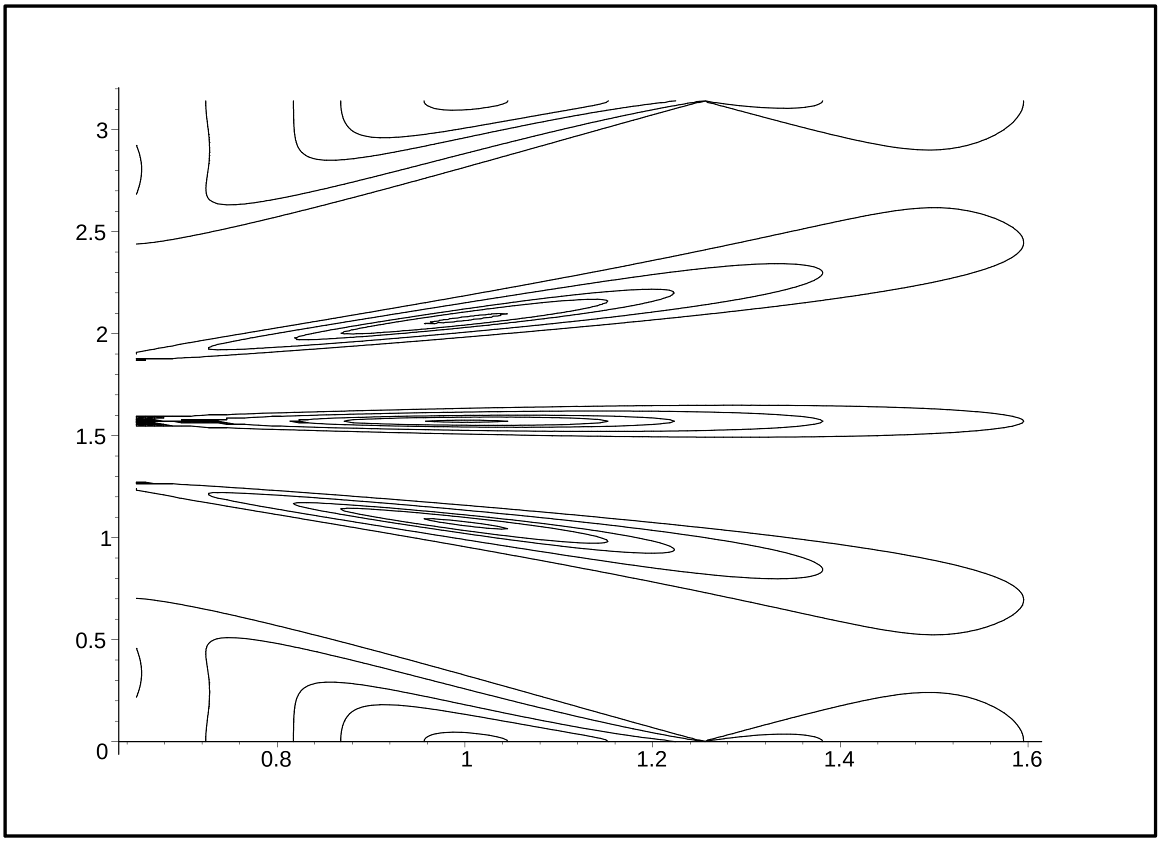

FIG. 4: Plot of $\mathcal{V} = 0$ for $A = 0.9, 0.95, 0.98, 0.99, 0.999$. Abscissa is x and ordinate is θ. The enclosed area decreases for increasing A and vanishes for $A \geq 1$.

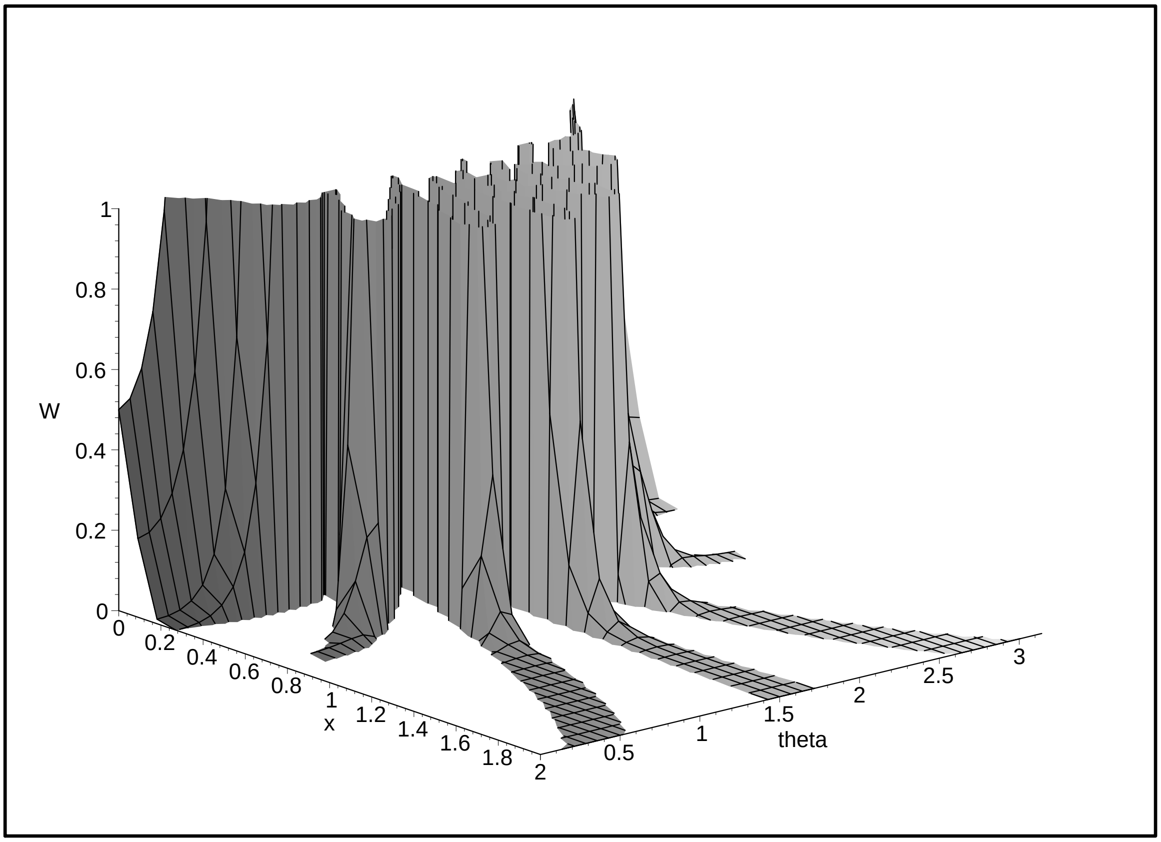

FIG. 5: Plot of $\mathcal{W}$ truncated at $\mathcal{W} = 0, +1$ for $A = 1/2$.

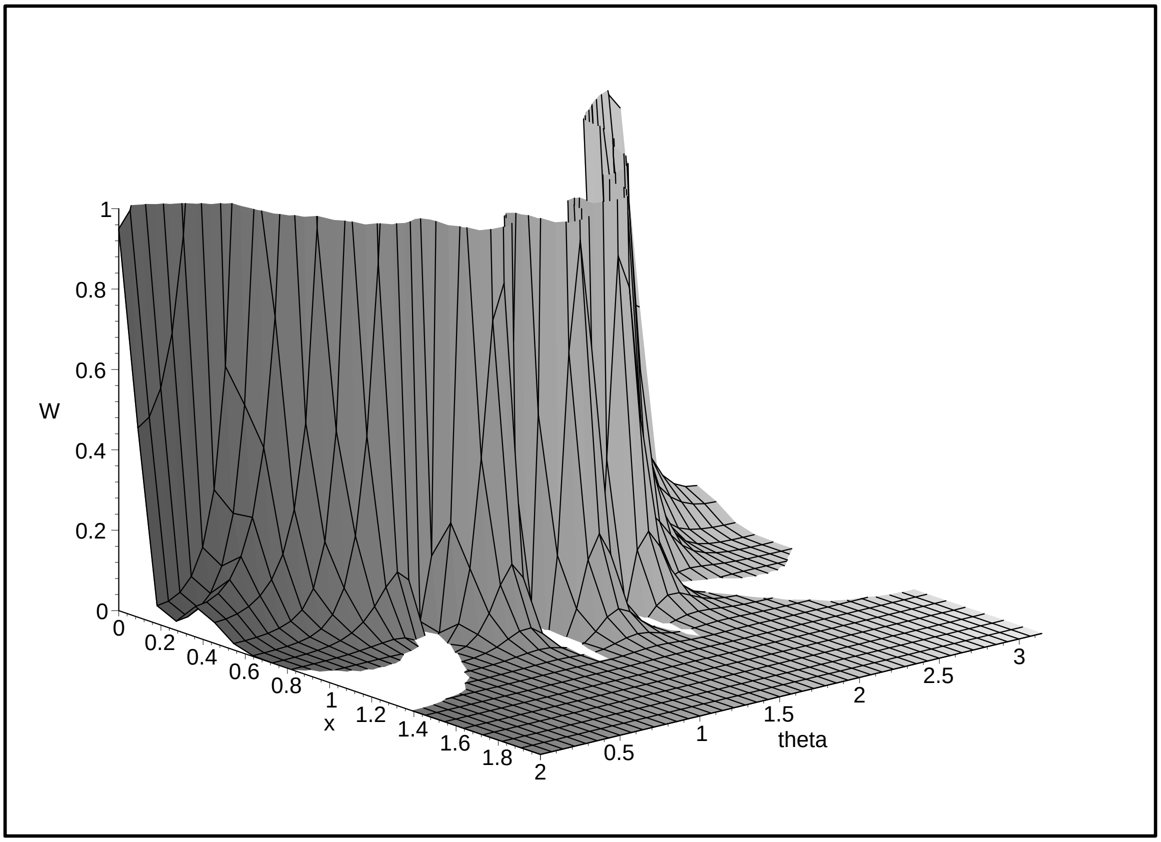

FIG. 6: Plot of $\mathcal{W}$ truncated at $\mathcal{W} = 0, +1$ for $A = 0.95$.

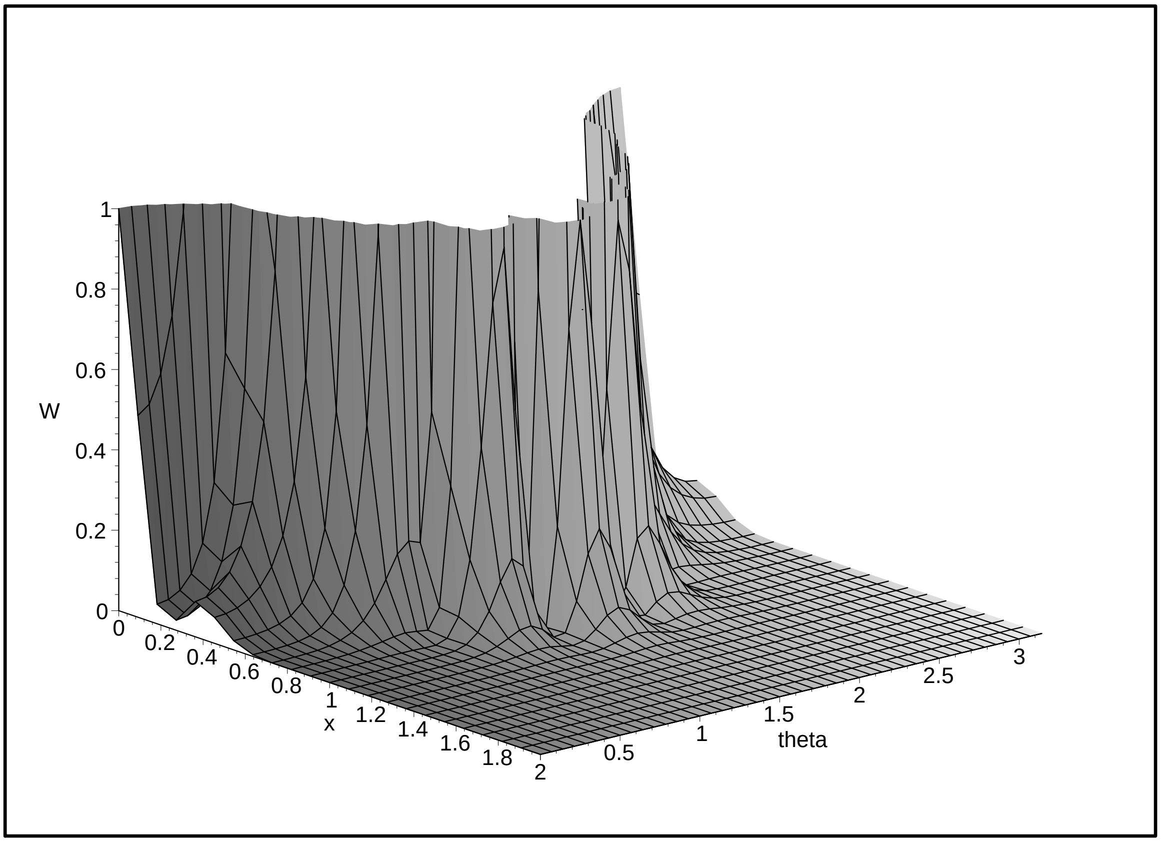

FIG. 7: Plot of $\mathcal{W}$ truncated at $\mathcal{W} = 0, +1$ for $A = 1$.

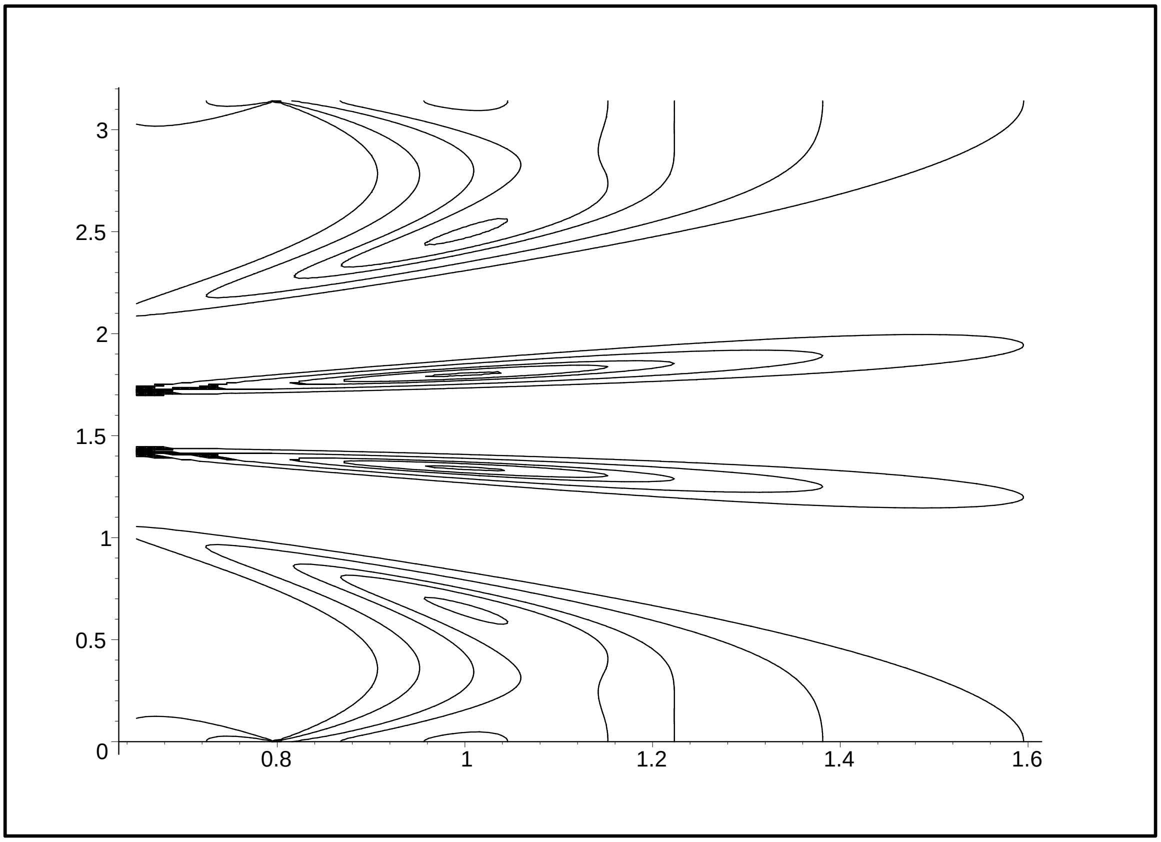

FIG. 8: Plot of $\mathcal{W} = 0$ for $A = 0.9, 0.95, 0.98, 0.99, 0.999$. Abscissa is x and ordinate is θ. The enclosed area decreases for increasing A and vanishes for $A \geq 1$.